THE

MATHEMATICAL MONTHLY.

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JUNE, 1859.

TERMS.

THE

MATHEMATICAL MONTHLY.

PRIZE PROBLEMS FOR STUDENTS.

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In a right-angled triangle, having given the difference between the base and perpendicular, and also the difference between the hypothenuse and base ; to construct the triangle geometrically.

II.

In a right-angled triangle, having given the sum of the base and perpendicular, also the sum of the hypothenuse and base; to construct the triangle geometrically.

III.

The four tangents, which are common to two circles which do not intersect, and are terminated at their points of respective contact, have their middle points on the radical axis of the two circles.

IV.

The external centres of similitude of three circles, taken successively two and two, all lie in one straight line ; and each of them is situated in a right line with two of the internal centres of similitude.

V.

Let two circles be touched respectively by a single straight line AA' in A and A', and by a single circle in $BB'C$ in B and B' ; if

the straight line and the circle touch in the same manner the two circles, the point C of the meeting of AB and $A'B'$ will lie on the circumference of the circle $BB'C$ and on the radical axis of the two other circles.

The solution of these problems must be received by the first of August, 1859.

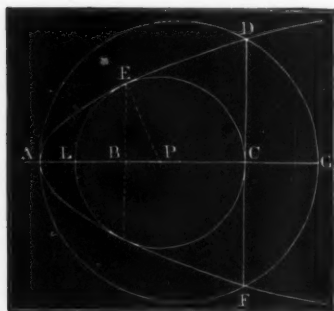### REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. V., Vol. I.

THE first Prize is awarded to ARTHUR W. WRIGHT, of the Senior Class in Yale College, New Haven, Ct.

The second Prize is awarded to GEORGE A. OSBORNE, Jr., of the Lawrence Scientific School, Cambridge, Mass.

PRIZE SOLUTION OF PROBLEM I.

"The abscissa and double ordinate of a segment of a common parabola are a and b, and the diameters of its circumscribed and inscribed circles D and d; to prove that $D + d = a + b$."

Let A be the origin of coördinates, and AG the axis of x, $AC = a$, and $DF = b$. The equation of the parabola is $y^2 = 2px$, and that of a circle having its centre on the axis of x is $(x - x_0)^2 + y^2 = r^2$. If this circle is tangent to DF, then we have $x_0 = AC - PC = a - r$, and its equation becomes $(x - a + r)^2 + y^2 = r^2$. Combining this with the equation of the parabola, we get $(x - a + r)^2 + 2px = r^2$; therefore, $x = a - r - p \pm \sqrt{(r + p)^2 - 2pa}$, which gives the abscissas of the points of intersection of the two curves. But when the circle becomes tangent to the parabola, these values of x must be equal,

and hence $\sqrt{(r+p)^2-2pa}=0$. $\therefore r=-p+\sqrt{2pa}=-p+\frac{1}{2}b$, since $2pa=\frac{1}{4}b^2$. Therefore $2r=d=b-2p$.

The equation of the circle circumscribing the segment is $y^2=2Rx-x^2$, which, combined with that of the parabola, gives $2px=2Rx-x^2$; and for $x=a$, $2R=D=a+2p$. Therefore $D+d=a+b$.

This solution is by ARTHUR W. WRIGHT.

NOTE. For E, the point of tangency, $x=a-r-p=AC-PC-BP$, and therefore $p=BP$; that is, the subnormal is constant and equal to the semi-parameter. Upon this property most of the solutions were based. Thus, in the right-angled triangle EBP, we have $r^2=y^2+p^2$; but $y^2=2px=2p(a-r-p)$; and therefore $r^2=2p(a-r-p)+p^2$. Hence $r=-p\pm\sqrt{2pa}=-p\pm\frac{1}{2}b$. And since $DC^2=AC\times CG$, or $\frac{1}{4}b^2=2pa=(2R-a)a$, therefore $2R=a+2p$.

It will be observed that there are two values of r; namely, $-p+\frac{1}{2}b$, and $-p-\frac{1}{2}b$, the first of which corresponds to the circle LEC. But a circle tangent to the parabola may be drawn tangent to DF on the side opposite to A; and if $x_0=a+r$, then $r=p\pm\frac{1}{2}b$; and as these values of r are the same as those above with changed signs, it follows that the same solution gives the numerical values of the radii of both the tangent circles. The second tangent circle was noticed by GEORGE A. OSBORNE, Jr.

When the inscribed circle is tangent to the parabola at A, the abscissa of the point of tangency is zero; that is $x=a-r-p=0$, or $a-\frac{1}{2}a-p=0$, or $a=2p$; which is the least value of a for which the problem holds. This limitation was noticed by GEORGE W. JONES, Jr., and ARTHUR W. WRIGHT.

PRIZE SOLUTION OF PROBLEM II.

"A great circle of the sphere passes through two given points; find the rectangular coördinates of its pole."

The equation of the sphere, referred to its centre as origin, is $x^2 + y^2 + z^2 = r^2$ (1). Let (x_1, y_1, z_1), (x_2, y_2, z_2) be the coördinates of the two given points, and (x, y, z) those of the required pole. Since the axis of the great circle, containing the given points, is perpendicular to its plane, it will be perpendicular to each of the radii containing these points; and also since $\left(\frac{x_1}{r}, \frac{y_1}{r}, \frac{z_1}{r}\right)$, $\left(\frac{x_2}{r}, \frac{y_2}{r}, \frac{z_2}{r}\right)$ and $\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$ are the cosines of the angles which the two radii and the axis of the circle make with the coördinate axes; we have

(2) $\quad \frac{x_1}{r} \cdot \frac{x}{r} + \frac{y_1}{r} \cdot \frac{y}{r} + \frac{z_1}{r} \cdot \frac{z}{r} = 0$, or $x x_1 + y y_1 + z z_1 = 0$;

(3) $\quad \frac{x_2}{r} \cdot \frac{x}{r} + \frac{y_2}{r} \cdot \frac{y}{r} + \frac{z_2}{r} \cdot \frac{z}{r} = 0$, or, $x x_2 + y y_2 + z z_2 = 0$.

From (2) and (3) we obtain

(4) $\quad y = \frac{x_1 z_2 - x_2 z_1}{z_2 y_2 - z_2 y_1} x$, and (5) $z = \frac{y_1 x_2 - y_2 x_1}{z_1 y_2 - z_2 y_1} x$,

which are the equations to the axis of the great circle. Combining (4) and (5) with (1), we have

$$x^2 + \left(\frac{x_1 z_2 - x_2 z_1}{z_1 y_2 - z_2 y_1}\right)^2 x^2 + \left(\frac{y_1 x_2 - y_2 x_1}{z_1 y_2 - z_2 y_1}\right)^2 x^2 = r^2;$$

or $\left[(z_1 y_2 - z_2 y_1)^2 + (x_1 z_2 - x_2 z_1)^2 + (y_1 x_2 - y_2 x_1)^2\right] x^2 = (z_1 y_2 - z_2 y_1)^2 r^2$.

The first member of this equation can be reduced by observing that

$(z_1 y_2 - z_2 y_1)^2 + (x_1 z_2 - x_2 z_1)^2 + (y_1 x_2 - y_2 x_1)^2 = (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)$

$- (x_1 x_2 + y_1 y_2 + z_1 z_2)^2 = r^4 - (x_1 x_2 + y_1 y_2 + z_1 z_2)^2 = r^4 - r^4 \cos^2 \varphi = r^4 \sin^2 \varphi$,

in which φ denotes the angle made by the two radii containing the given points.

Hence we have $x^2 r^4 \sin^2 \varphi = (z_1 y_2 - z_2 y_1)^2 r^2$; or $x = \pm \frac{z_1 y_2 - z_2 y_2}{r \sin \varphi}$.

Similarly it may be shown that $y = \pm \frac{x_1 z_2 - x_2 z_1}{r \sin \varphi}$, and $z = \pm \frac{y_1 x_2 - y_2 x_1}{r \sin \varphi}$.

This solution is by GEORGE A. OSBORNE, Jr.

PRIZE SOLUTION OF PROBLEM III.

" If the two sides of a movable right angle are always tangents to a given ellipse, its summit will describe a circle concentric with the ellipse, the radius of which is equal to the chord joining the extremities of the major and minor axes."

This is simply to find the locus of the intersection of pairs of tangents which are at right angles to each other. Let the equation of the given ellipse be $A^2 y^2 + B^2 x^2 = A^2 B^2$ (1), and α, β the coördinates of any point in the locus. The equation of a straight line passing through α, β is $y - \beta = m (x - \alpha)$ (2).

Let y be eliminated between (1) and (2), and the result arranged according to powers of x; then

$$(A^2 m^2 + B^2) x^2 + 2 A^2 m (\beta - m \alpha) x + A^2 (\beta^2 + \alpha^2 m^2 - 2 \beta \alpha m - B^2) = 0.$$

In order that (2) may be a tangent to (1), the values of x from this quadratic must be *equal*. Whence by the theory of equations

$$[2 A^2 m (\beta - m \alpha)]^2 = 4 A^2 (A^2 m^2 + B^2) (\beta^2 + \alpha^2 m^2 - 2 \beta \alpha m - B^2).$$

Developing this expression, reducing and arranging according to powers of m,

$$m^2 + \frac{2 \alpha \beta}{\alpha^2 - A^2} m + \frac{\beta^2 - B^2}{\alpha^2 - A^2} = 0.$$

The two values of m from this equation can only belong to the two tangents which can be drawn to the ellipse through α, β. Calling one value m, and the other m', the theory of equations gives the following relation :

$$m m' = \frac{\beta^2 - B^2}{A^2 - \alpha^2};$$

and, since the tangents are at right angles to each other, $m m' = - 1 = \frac{\beta^2 - B^2}{\alpha^2 - A^2}$. Whence $\alpha^2 + \beta^2 = A^2 + B^2$, and therefore the locus is a circle.

This solution is by GEORGE B. HICKS.

PRIZE SOLUTION OF PROBLEM IV.

" If a circle be described through the foci of an ellipse and any point in the con-

jugate axis produced; to prove that the right line joining that point and one of the points where the circle cuts the ellipse will be a tangent to the ellipse."

Let the circle $F'EF$, passing through any given point as E in the conjugate diameter of the ellipse $A'BA$, and also through its foci, meet the curve in the point P. Join PE', and from P draw PF and PF' to the foci F and F', and from E through P draw EP. Since EE' is perpendicular to AA', and $CF = CF'$, the arc FE' is equal to the arc $F'E'$, and therefore the angles FPE' and $E'PF'$ are equal. But EPE' is a right angle, since the arc EPE' is a semicircle. Hence the angles TPF and EPF' are equal, and therefore EPT is tangent to the ellipse.

This solution is by ARTHUR W. WRIGHT; and most of the others are essentially the same.

PRIZE SOLUTION OF PROBLEM V.

"If D represent any diameter of an ellipse, and P the parameter of D, to find when $D + P$ is the least, and when the greatest, possible."

Since the sum of the squares of any two conjugate diameters equals the sum of the squares of the axes, $D'^2 + D^2 = 4(A^2 + B^2)$. But $P = \dfrac{D'^2}{D}$, and $D + P = D + \dfrac{D'^2}{D} = \dfrac{D^2 + D'^2}{D} = \dfrac{4(A^2 + B^2)}{D}$. Hence, since $A^2 + B^2$ is constant, $D + P$ varies inversely as D, and is greatest when D is least, or $2B$, and least when D is greatest, or $2A$.

This is substantially the solution given by nearly all the competitors.

<div style="text-align:right">

JOSEPH WINLOCK.
CHAUNCEY WRIGHT.
TRUMAN HENRY SAFFORD.

</div>

CONSTRUCTION OF A PROBLEM.

By J. E. HILGARD, United States Coast Survey, Washington, D. C.

PROBLEM. *Given, an elliptical right cone, to construct the angle made with its axis by a plane which intersects it in a circle.*

The solution consists in describing a sphere from any point of the axis of the cone as centre, and tangent to the longest sides of the cone. This sphere will intersect the surface of the cone in two circles, the planes of which are determined by the intersections of the sphere with the shortest sides of the cone.

Let SAB be a section through the major axis of the base, and

SDE a section through the minor axis. From any point C of the axis inscribe in the angle ASB the circle LHG; then the lines FG, HK, joining the intersections of the circumference with the opposite sides of the section SED, will give the required angle with the axis.
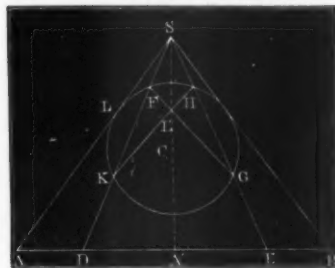
This elegant solution is due to Prof. ENGEL. Its demonstration is left to the student.

The method is of general application to surfaces of the second order having circular sections, such as the oblique elliptical cone, the ellipsoid of three axes, the elliptical and hyperbolical hyperboloids, &c.

QUESTION, BY MATTHEW COLLINS, B. A., DUBLIN, IRELAND.

IF we multiply a circulating decimal (pure or mixed) having m figures in its period by another such circulate having n figures in its period, prove that the product will be a circulate having $9\,mn$ figures in its period if m be prime to n. But if n be equal to m

or to a multiple of m, prove that the period in the product will then have $n(10^m-1)$ figures in it; if $m=4$ and $n=6$, prove that the period in the product will consist of $99 \times 6 \times 2$, or 1188 figures; also extend and generalize this latter part of this new and curious theorem.

PROPOSITIONS RELATING TO THE RIGHT-ANGLED TRIANGLE.

BY M. L. COMSTOCK,
Assistant Professor of Mathematics in Knox College, Galesburg, Illinois.

I send you a few additional propositions relating to right-angled triangles, which may be useful to those who require original demonstrations of their pupils.

1. In any right-angled triangle, if the circumference of a circle be drawn through the extremities of the hypothenuse and the centre of the inscribed circle; the diameter of the circle so drawn will be the diagonal of the square described on the hypothenuse, and its centre will be the circumference of the circle circumscribing the triangle.

2. If a circumference of a circle be drawn through both extremities of either leg and the centre of the inscribed circle, its centre will be on the circumference of the circumscribing circle.

3. If the lines which bisect the acute angles of a right-angled triangle be drawn, and produced to meet the circumference of the circumscribing circle, the rectangle of the parts intercepted between the angles and the opposite sides of the triangle is equal to four times the rectangle of the lines intercepted between the centre of the inscribed, and circumference of the circumscribing, circles.

4. Also the first-mentioned rectangle is equal to twice the rectangle of the lines intercepted between the acute angles and centre of the inscribed circle.

NOTE ON THE CYCLOID.

BY LEWIS R. GIBBES,
Professor of Mathematics in the College of Charleston, South Carolina.

In many elementary treatises on Mechanics, the student is informed that the cycloid is the curve of quickest descent under the restrictions usually given, but the demonstration of this property of the cycloid is supposed to be inaccessible to him, without a knowledge of the calculus of variations. The following geometrical demonstration will be valued by the student whose course of study does not extend so far, and perhaps may not be contemned by those who are familiar with all the resources of the calculus. We do not believe it possible to present the demonstration in a simpler form. We will first premise the three following propositions easily demonstrable.

Prop. 1. If two right-angled triangles have the same altitude, the product of the sum and difference of their hypothenuses is equal to the product of the sum and difference of their bases. Easily deducible from the well-known theorem of the relation of the sides of a right-angled triangle.

Prop. 2. The velocity at any point of descent in a curve is equal to the velocity due to the vertical fall or height; which velocity is proportional to the square root of that height. Given in the treatises.

Prop. 3. At any point in the cycloid,

Fig. 1.

$$\frac{\text{the increment of the ordinate,}}{\text{the increment of the curve} \times \sqrt{\text{axis} - \text{abscissa,}}}$$

is a constant quantity, or, what is the same, such quantities for any two points in the curve are equal; that is, in Fig. 1, $\dfrac{PS}{PQ\sqrt{PF}} = \dfrac{QT}{QR\sqrt{QG}}$ the origin of coördinates being at the vertex A.
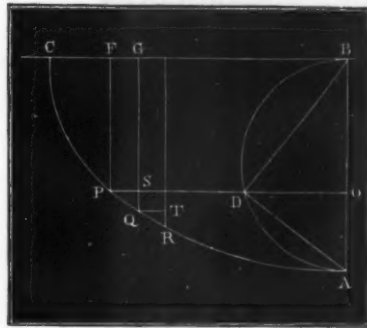
By the properties of the cycloid, the elementary triangle PQS, formed by the increment of the curve PQ, the increment of the abscissa QS, and the increment of the ordinate SP, is similar to the triangle DAO, and hence similar to BAD. $\therefore \dfrac{PS}{PQ} = \dfrac{BD}{BA} = \dfrac{\sqrt{BO}}{\sqrt{BA}} = \dfrac{\sqrt{PF}}{\sqrt{BA}}$, and $\dfrac{PS}{PQ\sqrt{PF}} = \dfrac{1}{\sqrt{BA}}$. In like manner, $\dfrac{QT}{QR\sqrt{QG}} = \dfrac{1}{\sqrt{BA}}$. Hence,

$$\frac{PS}{PQ\sqrt{PF}} = \frac{QT}{QR\sqrt{QG}}.$$

PROBLEM. *To find the path of quickest descent from one point to another not in the same vertical line, the particle being urged by the action of gravity, or of a uniformly accelerating force acting in a direction parallel to a given line.*

We will suppose the descent effected in a non-resisting medium, in the vertical plane passing through the two points; and we must seek the relation between the velocity at any point, and the increments of the curve, and of one of the coördinates of that point.

Let H and I, Fig. 2, be the two points, and $HPQRI$ the curve

Fig. 2.

of quickest descent from H to I; then, necessarily, the portion of the curve between any two points, as P and Q, or Q and R, will be the path of quickest descent between those points P and Q, or Q and R, in the curve; otherwise the whole curve HI could not be the path of quickest descent from H to I, for the velocity acquired on reaching R, and the remaining time through RI, which depends on this velocity, will be precisely the same by whatever path R be reached; so that the property of the whole curve belongs to every part of it.

Let PQR be an elementary portion of the curve; then, since the time is less through PQR than through any other paths PKR, PNR on either side, and is greater the further these paths recede

from PQR, it follows that there must be pairs of paths, one member on each side of PQR, in which the times of descent, though each longer than in PQR, are yet precisely equal. Let PKR and PNR be one such pair of paths, each member of the pair deviating infinitely little from PQR, and divided into two portions at K and N by the horizontal line LT, drawn so as to cut the curve into two equal (or unequal) portions at Q. These elementary portions may be regarded as straight lines described uniformly, PK and PN with the velocity acquired at P, KR and NR with the velocity acquired at K and N. By Proposition 2, the velocity at P is proportional to $\sqrt{PF}$, that at K and N to $\sqrt{QG}$; hence, since in uniform motion, time $= \frac{\text{space}}{\text{velocity}}$, and since the sum of the times through PN, NR is equal to the sum of the times through PK and KR, we have

$$\frac{PN}{\sqrt{PF}} + \frac{NR}{\sqrt{QG}} = \frac{PK}{\sqrt{PF}} + \frac{KR}{\sqrt{QG}}, \text{ or } \frac{PN - PK}{\sqrt{PF}} = \frac{KR - NR}{\sqrt{QG}}.$$

Now, to introduce the increments of one of the coördinates of the curve, multiply the numerator and denominator of each fraction by the sum of the quantities whose difference forms its numerator, and then, by Proposition 1, for the product of the sum of the hypotenuses of the elementary triangles PKL, PNL, KRT, NRT, substitute the product of the sum and difference of their bases, and we have $\frac{(LN + LK)\,KN}{(PN + PK)\sqrt{PF}} = \frac{(KT + NT)\,KN}{(KR + NR)\sqrt{QG}}$; dividing by NK, then $\frac{LN + LK}{(PN + PK)\sqrt{PF}} = \frac{KT + NT}{(KR + NR)\sqrt{QG}}$.

This equation holds good for every such pair of paths, however near to PQR, and is true when K and N coincide with Q; in that case $LN + LK = 2LQ = 2PS$, $PN + PK = 2PQ$, $KT + NT = 2QT$, and $KR + NR = 2QR$. Hence, substituting these quantities for their equals and dividing by 2, we have,

finally $\dfrac{PS}{PQ\sqrt{PF}} = \dfrac{QT}{QR\sqrt{QG}}$, which is precisely the property of the cycloid proved in Proposition 3. So that an arc of a cycloid is the brachistochrone, or curve of quickest descent, under the supposed conditions.

The points F and G in Fig. 1 are in the base of the cycloid, so that, in Fig. 2, *the base of the cycloid must pass through the point from which the descent begins;* this condition is often not distinctly mentioned. If the two points H and I be nearly on a level, the arc may be almost the whole cycloid.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

SECTION III.

ON THE MOTIONS AND FIGURE OF A SMALL CIRCULAR PORTION OF FLUID ON THE EARTH'S SURFACE.

24. WE shall, in this case, suppose that α is a function of
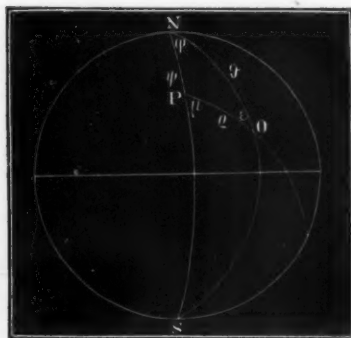
Fig. 2

the distance from the centre of the fluid. It will be more convenient, therefore, to express our general equations (20) in terms of other polar coördinates, of which the pole P, Fig. 2, does not correspond with the pole of the earth. Regarding the earth as a perfect sphere, let ψ be the distance in arc of the new pole P from the pole of the earth; also let

ϱ be the distance in arc from the pole P,

μ the angle $S\,P\,O$ between ϱ and the meridian,

ε the alternate angle $N\,O\,P$.

If in equations (20) we put $n=0$, they become the equations of horizontal motions in the case in which the earth has no rotary motion, and the pole of the coördinates can in this case be assumed at pleasure. Hence, when the earth has no rotation, by putting ϱ for ϑ, and μ for φ, we have

$$
\begin{aligned}
\text{(33)} \quad & g\,D_\rho h = r^2 \sin\varrho \cos\varrho\,(D_t\varphi)^2 - r^2\,D_t^2\varrho - g\,h\,D_\rho \log\alpha, \\
& g\,D_\mu h = -2\,r^2 \sin\varrho \cos\varrho\,D_t\varrho\,D_t\mu - r^2 \sin^2\varrho\,D_t^2\mu.
\end{aligned}
$$

When the earth has a rotation, we must add to the second members of these equations respectively the terms $D_\rho F$, and $D_\mu F$, in which F is the part of P, equation (10) depending upon the earth's rotation, and must satisfy the following equations,

$$
\begin{aligned}
D_\theta F &= 2\,r^2\,n \sin\vartheta \cos\vartheta\,D_t\varphi, \\
D_\phi F &= -2\,r^2\,n \sin\vartheta \cos\vartheta\,D_t\vartheta.
\end{aligned}
$$

Since ϑ and φ are functions of ϱ and μ, we must put

$$
\begin{aligned}
D_\rho F &= D_\theta F . D_\rho \vartheta + D_\phi F . D_\rho \varphi, \\
D_\mu F &= D_\theta F . D_\mu \vartheta + D_\phi F . D_\mu \varphi.
\end{aligned}
$$

Hence, substituting the preceding values of $D_\theta F$ and $D_\phi F$, we get

$$
\begin{aligned}
\text{(34)} \quad & D_\rho F = 2\,r^2\,n \sin\vartheta \cos\vartheta\,(D_t\varphi\,D_\rho\vartheta - D_t\vartheta\,.\,D_\rho\varphi), \\
& D_\mu F = 2\,r^2\,n \sin\vartheta \cos\vartheta\,(D_t\varphi\,D_\mu\vartheta - D_t\vartheta\,.\,D_\mu\varphi).
\end{aligned}
$$

Now, from the relations of the different parts of a spherical triangle, we have

$$
\begin{aligned}
\text{(35)} \quad & \cos\vartheta = \cos\psi \cos\varrho - \sin\psi \sin\varrho \cos\mu, \\
& \cot\varphi = \frac{\sin\psi \cos\varrho + \cos\psi \sin\varrho \cos\mu}{\sin\psi \sin\mu}.
\end{aligned}
$$

Hence, taking the derivatives and reducing, we get

$$
D_\rho \vartheta = \frac{\cos\psi \sin\varrho + \sin\psi \cos\varrho \cos\mu}{\sin\vartheta} = \cos\varepsilon,
$$

$$D_\mu \theta = -\frac{\sin \psi \sin \varrho \sin \mu}{\sin \theta} = -\sin \varrho \sin \varepsilon,$$

$$D_\rho \varphi = \frac{\sin^2 \varphi}{\sin \psi \sin \mu} = \frac{\sin \varepsilon}{\sin \theta},$$

$$D_\mu \varphi = \frac{\cos \psi \sin \varrho + \sin \psi \cos \varrho \cos \mu}{\sin^2 \mu \sin \varrho} \sin^2 \varphi = \frac{\sin \varphi \cos \varepsilon}{\sin \theta},$$

$$D_t \theta = D_\rho \theta . D_t \varrho + D_\mu \theta . D_t \mu = \cos \varepsilon \, D_t \varrho - \sin \varrho \sin \varepsilon \, D_t \mu,$$

$$D_t \varphi = D_\rho \varphi . D_t \varrho + D_\mu \varphi . D_t \mu = \frac{\sin \varepsilon}{\sin \theta} D_t \varrho + \frac{\sin \varrho \cos \varepsilon}{\sin \theta} D_t \mu.$$

These values being substituted in (34), we get

(36)
$$D_\rho F = 2 \, r^2 n \sin \varrho \cos \theta \, D_t \mu,$$
$$D_\mu F = -2 \, r^2 n \sin \varrho \cos \theta \, D_t \varrho.$$

If we add these values of $D_\rho F$ and $D_\mu F$ respectively to the second members of (33), we get for the equations of motion, in terms of ϱ and μ, when the earth has a rotation,

(37)
$$g D_\rho h = r^2 \sin \varrho \, (2n \cos \theta + D_t \mu \cos \varrho) \, D_t \mu - r^2 D_t^2 \varrho - g h D_\rho \log \alpha,$$
$$g D_\mu h = -2 \, r^2 \sin \varrho \, (n \cos \theta + D_t \mu \cos \varrho) \, D_t \varrho - r^2 \sin^2 \varrho \, D_t^2 \mu,$$

in which $\cos \theta$ has the value in terms of ϱ and μ, in the first of (35).

25. When $\sin \varrho$ is so small that the last term of the value of $\cos \theta$ may be neglected in comparison with the first, we have $\cos \theta = \cos \psi \cos \varrho$, which being substituted in the last equations, they become

(38)
$$g D_\rho h = r^2 \sin \varrho \cos \varrho \, (2n \cos \psi + D_t \mu) \, D_t \mu - r^2 D_t^2 \varrho - g h D_\rho \log \alpha,$$
$$g D_\mu h = -2 \, r^2 \sin \varrho \cos \varrho \, (n \cos \psi + D_t \mu) \, D_t \varrho - r^2 \sin^2 \varrho \, D_t^2 \mu.$$

These equations are similar to equations (20), having ϱ and μ instead of θ and φ, and, instead of n, having $n \cos \psi$, which is the earth's angular velocity of rotation around the axis, corresponding with the pole P (PEIRCE's *Analytical Mechanics*, § 25). Hence we can treat these equations precisely as equations (20) in the last section, and, instead of (21), we get

(39) $$r^2 \sin^2 \varrho \, (n \cos \psi + D_t \mu) = c,$$

and, instead of (22), we get

(40) $$\int_m r^2 \sin^2 \varrho \, (n \cos \psi + D_t \mu) = \int_m c = C m.$$

On account of the term which has been neglected in the value of $\cos \vartheta$, these equations cannot be used for large values of ϱ, and hence we may put $\sin \varrho = \varrho$. Let

$s = R \varrho$ be the lineal distance from the centre,

s' be the value of s at the external part of the fluid,

u be the initial value of $D_t \mu$.

The last equation then gives, putting R for r,

$$C m = \int_m s^2 \, (n \cos \psi + u),$$

$$= \int_N^l \int_\mu^{2\pi} \int_s^{s'} k \, s^3 \, (n \cos \psi + u),$$

$$= \tfrac{1}{2} s'^2 m \, (n \cos \psi + u'),$$

in which

$$u' = \frac{2}{s'^2 m} \int_N^l \int_\mu^{2\pi} \int_s^{s'} k \, s^3 \, u.$$

Hence,

(41) $$C = \tfrac{1}{2} s'^2 \, (n \cos \psi + u').$$

In the preceding integration k is regarded as a constant. When, by the mutual action of the different strata upon each another, $D_t \mu$ becomes the same at all altitudes at the same distance from the centre P, c becomes equal to C, and equation (39) then gives

(42) $$D_t \mu = \frac{C}{R^2 \sin^2 \varrho} - n \cos \psi = \frac{s'^2 \, (n \cos \psi + u')}{2 \, s^2} - n \cos \psi.$$

26. If we suppose the initial state of the fluid to be that of rest relative to the earth's surface, the last equation becomes

(43) $$D_t \mu = \left(\frac{s'^2}{2 \, s^2} - 1 \right) n \cos \psi.$$

Substituting this value of $D_t \mu$ in the first of equations (38), it becomes, by putting R for r and $\cos \varrho = 1$,

$$(44) \quad g\, D_s h = n^2 \cos^2 \psi \left(\frac{s'^4}{4\,s^3} - s \right) - D_t^2 s - g\, h\, D_s \log \alpha.$$

This equation is similar to (27), and, like it, can only be satisfied by means of an interchanging motion between the internal and external part of the fluid; and the remarks following that equation in § 15 are also applicable to this.

27. By omitting the last two terms in the preceding equation, as was done in equation (25), (§ 16), we get by integration,

$$2\, g\, h = - n^2 \cos^2 \psi \left(\frac{s'^4}{4\, s^2} + s^2 \right) + C.$$

Hence, eliminating C,

$$(45) \quad h = h' + \frac{n^2 \cos^2 \psi}{2\, g} \left(\tfrac{1}{4}\, s'^2 - \frac{s'^4}{4\, s^2} - s^2 \right).$$

Since one of the negative terms in this value of h has s in the denominator, it must become equal 0 towards the centre where s vanishes. Hence *the fluid, however deep it may be at the external part, cannot exist at the centre.*

28. If we put s_0 for the value of s where $h = 0$, the last equation gives

$$(46) \quad 0 = h' + \frac{n^2 \cos^2 \psi}{2\, g} \left(\tfrac{1}{4}\, s'^2 - \frac{s'^4}{s_0^2} - s_0^2 \right),$$

from which we obtain s_0 for any assumed value of h'.

Since s_0 is very small, the terms $\tfrac{1}{4}\, s'^2$ and $-\, s_0^2$ may generally be omitted in the last equation, and it then becomes

$$(47) \quad s_0 = \frac{n \cos \psi\, s'^2}{\sqrt{2\, g\, h'}}.$$

If we put s_1 for s where h is a maximum, equation (44) gives, by putting $D_s h = 0$, and neglecting the last two terms,

$$(48) \quad s_1 = \frac{s'}{\sqrt{2}}.$$

Equation (45) determines the figure of the surface of the fluid,

Fig. 3.
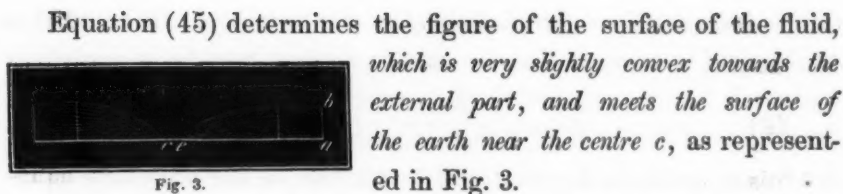
which is very slightly convex towards the external part, and meets the surface of the earth near the centre c, as represented in Fig. 3.

If we assume h', or ab, Fig. 3, equal 5 miles, and $ac = 100$ miles, equation (49) gives $ce = 2$ miles nearly.

29. Equation (43) gives the angular velocity of gyration, which must be very great near the centre, where s is small.

Putting $D_t \mu = 0$, it gives

$$(49) \qquad s = \frac{s'}{\sqrt{2}} = s_1.$$

Hence, *at the distance of s_1, which is the distance of the maximum of* h, *there is no gyratory motion.*

In the northern hemisphere, where $\cos \psi$ is positive, if $s < s'$, $D_t \mu$ is positive, but if $s > s'$, it is negative. Hence *the inner part of the fluid gyrates from right to left, but the external part from left to right,* as represented in Fig. 4. In the southern hemisphere, where $\cos \psi$ is negative, the gyrations are the reverse.
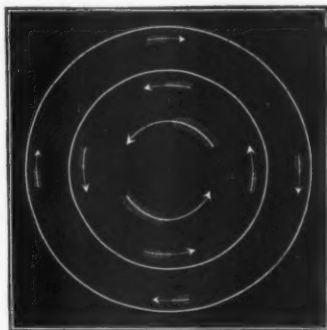
Fig. 4.

30. If the fluid is of uniform density, and every part gyrates with the same angular velocity u, it satisfies equations (38) by satisfying the following equation:

$$g\, D_s\, h = 2\, s\, u\, n \cos \psi + s\, u^2,$$

since all the other terms vanish; and this motion also satisfies the condition of continuity. By integrating, we get

$$(50) \qquad g\, h = \tfrac{1}{2}\, s^2 u\, (2\, n \cos \psi + u) + C.$$

This is the equation of a parabola. Hence *the surface of the fluid,*

relative to the earth's surface, is the surface of a paraboloid. If the portion of the fluid is so small that the earth's surface may be regarded as a plane, it becomes absolutely the surface of a paraboloid; and when the angular velocity of gyration is great in comparison with that of the earth's rotation, $2\,n \cos \psi$ may be omitted, in the preceding equation, in connection with u.

If $u = -\,2\,n \cos \psi$, or $u = 0$, h is constant, and then the surface of the fluid is a level surface. If u is negative and less than $2\,n \cos \psi$, the surface is convex; in all other cases it is concave.

31. If the whole of a gyrating mass of fluid has a tendency to move in the direction of the meridian with a force V, if we regard the forces which act upon each part of the fluid in the directions of the meridians as parallel, we have, using R for r,

$$V = m\, D_t^2\, \psi = \frac{1}{R} \int_m D_\theta^\prime P.$$

The error arising from regarding the forces in the directions of the meridians parallel is of the second order of their deviation from parallelism, and consequently very small, unless the lateral extent of the fluid is very great.

From the last equation and the second of equations (9), omitting the term containing $D_t r$ as a factor, since it can produce no sensible effect, we get

$$V = \int_m [-\,R\, D_t^2\, \theta + R \sin \theta \cos \theta\, (2\,n + D_t \varphi)\, D_t \varphi].$$

If in this equation we substitute for $D_t \varphi$ its value in §24, and for $D_t^2 \theta$ its value derived from that of $D_t \theta$ in the same section, and also for $\cos \theta$ its value in the first of equations (35), putting $\varepsilon = \mu$, since the meridians are regarded as parallel, and omitting all terms which give 0 by integration, we get

(51)
$$\begin{aligned} R\,V &= -\,2\,n \sin \psi \int_m s^2 \cos^2 \mu\, D_t \mu, \\ &= -\,n \sin \psi \int_m s^2 D_t \mu. \end{aligned}$$

If $D_t \mu$, the angular velocity of gyration, is positive, V is negative; but positive, if $D_t \mu$ is negative. Hence *if the fluid gyrates from right to left, the whole mass has a tendency to move towards the north; but if from left to right, towards the south.*

If every part of a cylindrical mass having its axis of revolution vertical has the same angular velocity of gyration as in the case of solids, calling this velocity u, the preceding equation gives for the accelerating force in the direction of the meridian,

$$(52) \qquad \frac{V}{m} = -\frac{s'^2 u n \sin \psi}{2 R} = -\frac{s'^2 u \sin \psi}{2 R^2 n} \times R n^2,$$

$$= -\frac{s'^2 u \sin \psi}{2 R^2 n} \times \frac{g}{289} = -\frac{g}{578} \cdot \frac{u \sin \psi}{n} \cdot \frac{s'^2}{R^2}.$$

32. If a body move in the direction of ϱ or s with a velocity $v = D_t s$, and p be the direction of a perpendicular to it on the left, we obtain from the last of equations (36) for the deflecting force in the direction of p, arising from the earth's rotation,

$$D_p F = \frac{D_\mu F}{R \sin \varrho} = -2 R n \cos \theta \, D_t \varrho,$$

$$(53) \qquad = -2 n \cos \theta \, D_t s = -\frac{2 \cos \theta \, D_t s}{R n} \times R n^2,$$

$$= -\frac{2 \cos \theta \, D_t s}{R n} \times \frac{g}{289} = -\frac{2 g v \cos \theta}{289 R n}.$$

This force is negative in the northern hemisphere, and positive in the southern. Hence *in whatever direction a body moves on the surface of the earth, there is a force arising from the earth's rotation, which deflects it to the right in the northern hemisphere, but to the left in the southern.* This is an extension of the principle upon which the theory of the trade winds is based, and which has been heretofore supposed to be true only of bodies moving in the direction of the meridian.

RESEARCHES IN THE MATHEMATICAL THEORY OF MUSIC.

By Truman Henry Safford, Cambridge, Mass.

1. Before entering directly upon the subject, it may not be inexpedient to premise some things concerning a few of the books upon it. The first writer whose works I have seen is Claudius Ptolemæus, the astronomer of the second century. His work " Harmonics "* is a treatise of the musical scale ; what we, in these days, should rather call a work on the theory of Melody. Parts of it seem quite applicable to the music of the present time, and part, to be theoretical only in the sense of an exposition of an untrue hypothesis. It is quite noticeable in the book, that the author uses the same argument to disprove what are now well-known facts that is often now urged against some views which are already held by many musicians, and which I may endeavor to support by mathematical reasoning. Ptolemy denies that the delicacy of the human ear is such that it can distinguish certain variations of tone ; he does not pay any regard to those musicians (the disciples of Aristoxenus) who controverted his theory by the evidence of their senses. He evidently had not such an ear for music as his opponents ; and his numerical system, if followed out, would have led to intolerable discords.

Aristoxenus seems to have held much the same view that is now entertained concerning the musical scale. Ptolemæus, on the contrary, maintained that the interval of a fourth (as we call it now) must be divided into separate steps, by any " super-particular ratios." That is, any system of three mixed numbers, each of the form $1 + \frac{1}{n}$, (n being a whole number) whose product should be $\frac{4}{3}$, would

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* Κλανδίου Πτολεμαίου ἁρμονικῶν βιβλία γ'.

represent in its component parts separate musical intervals, into which a "fourth" could be divided. That this system is utterly opposed to music is known to all conversant with the subject.

The third volume of "WALLIS's Opera Mathematica"* contains PTOLEMY's book, and those of two other writers, much later,— all in Greek, with a Latin translation by WALLIS. A good deal is said in them about the "church tones," scales differing in some respects from our modern major and minor, and professedly lineal descendants of the Dorian, Phrygian, &c., modes of the Greeks.

All these throw light on our modern musical system only incidentally and, as it were, by contrast; but there is one thing mentioned in them theoretically, which actually occurs in modern practice. It is evident, that, if we raise $\frac{3}{2}$ to the third power, we shall get $\frac{27}{8}$, quite a near approximation to $\frac{10}{3}$. Now, in tuning a violin, the fourth string—which should, theoretically, give the note vibrating in $\frac{10}{3}$ the time occupied by that produced by the lowest string (both being open, as it is termed, neither being pressed with the finger while the bow is drawn across them)— really is tuned so as to give the one vibrating in $\frac{27}{8}$ the time of the lowest; and this is effected by the same process represented arithmetically above. The lowest g string is made the foundation; the next or $\bar{d}$ string is made to vibrate in $\frac{3}{2}$ the time of that; the $\bar{a}$ string, again, in $\frac{3}{2}$ the time of that; and, finally, the $\bar{e}$ string, or highest, in $\frac{3}{2}$ the time of the latter.

The old writers used theoretically this mode of procedure in all cases; that is, the fraction $\frac{19}{27}$ takes the place of $\frac{4}{5}$, and $\frac{81}{64}$ the place of $\frac{5}{4}$, in their respective harmonic relations. The pure or nearly pure *major third*, indispensable in our day, was ignored entirely.

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* Πορφυρίου εἰς τὰ ἁρμονικὰ Πτολεμαίου ὑπόμνημα. Μανουὴλ Βρυεννίου ἁρμονικά.

An interesting work of the last century is one by MARPURG,* a distinguished musician. This is a little book on the Theory of Temperament (a subject relating to the means employed to modify the fractions ½ and ¾, or the like, so that, practically, they shall be equal, — which indeed is the great difficulty in the whole matter.) MARPURG, however, writes for people who do not apparently know how to extract roots, and have not seen a table of logarithms. For his treatment of numbers is laborious in the extreme, and he has to teach arithmetic as he goes along. A little knowledge of mathematics would have saved half the pages in his book.

The great EULER,† who has anticipated many things supposed to be late discoveries, seems to have been the first to remark, that musical intervals had something logarithmic in their nature. For as an interval (the difference of pitch of two tones) depends only upon the ratio of frequency of their vibrations; as an interval is added to another interval by multiplying the ratio of vibrations corresponding to one with that corresponding to the other; and as, in consequence, intervals are multiplied and divided by raising to powers and extracting roots of their respective ratios, — we see at once that an interval is best represented by the logarithm of the ratio of times of vibrations of the two notes comprising it.

As, however, the unit of musical intervals is the *octave*, whose times of vibration are as 1 to 2, the logarithms whose base is 2 are the fundamentals.

Much of EULER's "Tentamen" is, however, rather impracticable — too speculative. Were his authority in such matters not so great,

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* Anfangsgründe der Theoretischen Musik.

† Tentamen novæ theoriæ Musicæ de certissimis Harmonicæ Principiis dilucide expositæ actore LEONHARDO EULER. Petropoli, ex typographia Academiæ Scientiarum, 1739.

it might perhaps be said, that he had missed the point of the whole subject.

SMITH's Harmonics,* a book published, about a century ago, by a Cambridge mathematician and theologian, is very valuable even at the present day; and may be considered as an indispensable help in the study of a portion of our subject.

Three articles, by Prof. M. W. DROBISCH,† which I have seen, are apparently very good. The difficulties I find with them are, that he almost totally ignores the music of the past age, and holds to that of the present day as the only true development of the art. But one of the greatest German composers — JOHN SEBASTIAN BACH — thought otherwise; and although but of a hundred years' standing, and just beginning to be appreciated, he yet wrote much music in the ancient " church tones."

DROBISCH's other fault seems to be, an attempt to apply " Least Squares " without consideration of " weights." This problem is presented to him: " Certain notes being required to do double duty, to serve in different and utterly distinct chords, which can only do so by virtue of such relations as that between $\frac{3}{5}$ and $\frac{4}{7}$, notes being permitted to sound together which are really only approximately concordant, to determine those notes in accordance with the principles of the method of least squares." Now an approximation to the ratio $\frac{4}{5}$ (perhaps .802) will correspond to two

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* Harmonics, or the Philosophy of Musical Sounds. By ROBERT SMITH, D. D., F. R. S., and Master of Trinity College. Second Edition. London, 1759.

† Uber die wissenschaftliche Bestimmung der musikalischen Temperatur, von MORITZ WILHELM DROBISCH, Professor, etc. Leipzig. (From Poggendorff's Annalen der Physik und Chemie, 1853, No. 11, pp. 353–388.)

Ueber Musikalische Tonbestimmung und Temperatur, von M. W. DROBISCH. (Saxon Translations, Vol. IV., p. 1, Leipzig, 1855.)

Ueber Musikalische Tonverhältnisse. (Saxon Translations, Vol. V., p. 1.)

notes sounding much more agreeably together, than two notes whose ratios are as nearly $= \frac{2}{3}$, namely, .6686. Indeed, as DROBISCH himself remarks in another place, the ear can distinguish slighter variations from purity of concord in the latter case, than in the former; and he has himself determined the ratio of such variations. This fact furnishes us with the means of determining, or at least acknowledging, the existence of weights (in the least-square sense of the term). But DROBISCH solves the equations which he obtains, putting all his weights $= 1$. More may be said about this in the proper place.

A good idea of the "Church Modes," so called, can be obtained from a clever little book on that subject by CHARLES CHILD SPENCER.*

Recently Prof. DEMORGAN is said to have written a memoir, which I have not yet seen. I presume, from an abstract of its contents, that it relates chiefly to practical matters; is for organ builders, etc.

Mr. H. W. POOLE,† in his Essay on "Perfect Intonation, and the Euharmonic Organ," introduces some novel ideas, and some which will in future be put in practice. The "Euharmonic Organ" of Messrs. ALLEY and POOLE is practically but little used, and may continue so as long as the organ music of the present day is played.

It may become necessary to cite other works than those above mentioned, as this short list is far from a complete one, even of the few books accessible in America.

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* "Concise Explanation of the Church Modes." London, Novello.
† Silliman's Journal, New Series, Vol. IX., pp. 68, 199.

𝔐athematical 𝔐onthly 𝔑otices.

ON THE STABILITY OF THE MOTIONS OF THE RINGS OF SATURN.

WE have just received an Essay, by J. CLERK MAXWELL, M. A., Late Fellow of Trinity College, Cambridge, and Professor of Natural Philosophy in the Marischal College and University of Aberdeen, which obtained the "Adams Prize," in the University of Cambridge. The subject for the Prize is stated in the following words : —

"*The Motions of Saturn's Rings.* The Problem may be treated on the supposition that the system of rings is exactly, or very approximately, concentric with Saturn, and symmetrically disposed about the plane of his equator ; and different hypotheses may be made respecting the physical constitution of the rings. It may be supposed (1) that they are rigid ; (2) that they are fluid, or in part aeriform ; (3) that they consist of masses of matter not mutually coherent. The question will be considered to be answered by ascertaining on these hypotheses severally, whether the conditions of mechanical stability are satisfied by the mutual attractions and motions of the Planet and the rings.

"It is desirable that an attempt should also be made to determine on which of the above hypotheses, the appearance both of the bright rings and the recently discovered dark ring, may be most satisfactorily explained ; and to indicate any causes to which a change of form, such as is supposed from a comparison of modern with earlier observations to have taken place, may be attributed."

LAPLACE investigated the motions of circular homogeneous solid rings, and found that they are in a state of instability, and would, therefore, finally fall upon the Planet and be destroyed. "Hence," he says, "it follows, that the separate rings which surround the body of Saturn are *irregular solids*, of unequal widths in the different parts of their circumferences ; so that their centres of gravity do not coincide with their centres of figure. These centres of gravity may be considered as so many satellites, which move about the centre of Saturn, at distances depending on the inequalities of the parts of each ring, and with velocities of rotation equal to those of their respective rings." (BOWDITCH's Translation.)

MR. MAXWELL, in his preface, states the conclusion thus : "If the rings were solid and uniform, their motion would be unstable, and they would be destroyed. But they are not destroyed, and their motion is stable ; therefore they are either not uniform or not solid." It will be seen that this latter alternative of this conclusion, although legitimate, was not considered by LAPLACE ; nor was it seriously entertained till the year 1851. In GOULD's Astronomical Journal, Vol. II., No. 1, May 2d, 1851, we find an article "On the Rings of Saturn," by Prof. G. P. BOND, Director of the Observatory of Harvard College, in which the author first treats the question of the multiple divisions of the ring historically, and concludes that the various changes which have been observed are most naturally and easily explained upon the supposition that they are fluid.

He then proceeds to show that "there are considerations to be drawn from the state of the forces acting on the rings which favor the hypothesis." On the supposition of a single solid ring, taking the most probable determinations of the necessary data, he finds "that it will be necessary to increase its attractive force by sixty times its probable value, in order to retain its particles on its surface." The next supposition is a single division into two equal rings ; and by

giving each ring such time of rotation as will retain particles on its middle from leaving their place, it is found that the resulting widths of the rings are entirely too great. It is then necessary to suppose a larger number of rings, having different times of rotation, but still with intervals sufficiently small to give the requisite amount of reflecting surface; but the same analysis shows that even this condition is insufficient. LAPLACE proved that these rings could not be uniform homogeneous solids; but if they are irregular solids, then their centres of gravity must revolve about the Planet, and if their inequalities are large enough to oppose their tendency to fall upon the body of the Planet, their mutual disturbance must be so great as to render a collision of the rings very probable, if not wholly unavoidable. The foregoing is a brief outline of the argument from which the author infers " that the whole ring is in a fluid state, or at least does not cohere strongly." " Finally," he says, " a fluid ring, symmetrical in its dimensions, is not of necessity in a state of unstable equilibrium with reference either to Saturn or the other rings."

In GOULD's Journal, Vol. II., No. 3, June 16th, 1851, we find an article by Prof. PEIRCE, " On the constitution of Saturn's Rings," giving the results of his analysis, " based on purely mechanical considerations," of which the following are the condensed statements : —

1. " I maintain, unconditionally, that *there is no conceivable form of irregularity and no com-bination of irregularities, consistent with an actual ring, which would serve to retain it permanently about the primary, if it were solid.*" In referring to different cases of irregularity, he says, " In any case, the result is essentially the same, — that they will not permanently support the ring ; that a solid ring would soon be destroyed ; and that *Saturn's* ring must, therefore, be fluid. It consists, in short, of a stream, or rather streams, of a fluid somewhat denser than water, flowing around the Planet."

2. " Even in the case of a fluid ring, the motion of its centre of gravity is not controlled by the primary."

3. " The power which sustains the centre of gravity of *Saturn's* ring is not, then, to be sought in the planet itself, but in his satellites."

4. " It follows, then, that no planet can have a ring, unless it is surrounded by a sufficient number of properly arranged satellites."

Next, GOULD's Journal, Vol. IV., No. 14, September 5th, 1855, contains the beginning of a paper by Prof. PEIRCE, " On the Adams Prize Problem for 1856," in which the discussion is divided into three cases of solid, fluid, and discontinuous. Under the head of a solid ring, we find the conclusion stated thus : "*The conditions of the permanence of a solid ring are then necessarily subject to an unstable element ; and they are therefore unstable, so that the solid ring must be excluded from any physical theory which rests upon a firm basis.*"

In the second hypothesis of a fluid ring, the conclusion is, that " *The fluid ring cannot then be regarded as one of real permanence without the aid of foreign support ; although the action of the primary is not positively destructive to this, as it is to the solid ring.*"

We next come to Mr. MAXWELL's Essay, and shall simply give the results of his analysis : —

1. A RIGID RING. " The result of this theory of a rigid ring shows not only that a perfectly uniform ring cannot revolve permanently about the planet, but that the irregularity of a permanently revolving ring must be a very observable quantity, the distance between the centre of the ring and its centre of gravity being between . 8158 and . 8279 of the radius. As there is no appearance about the rings justifying a belief in so great an irregularity, the theory of the solidity of the rings becomes very improbable."

2. A RING OF EQUAL SATELLITES. " We next examined the motion of a ring of equal satellites, and found that if the mass of the planet is sufficient, any disturbances produced in

the arrangement of the ring will be propagated round it in the form of waves, and will not introduce dangerous confusion. If the satellites are unequal, the propagation of the waves will no longer be regular, but disturbances of the ring will in this, as in the former case, produce only waves, and not growing confusion. Supposing the ring to consist, not of a single row of large satellites, but of a cloud of evenly distributed unconnected particles, we found that such a cloud must have a very small density in order to be permanent, and that this is inconsistent with its outer and inner parts moving with the same angular velocity. Supposing the ring to be fluid and continuous, we found that it will be necessarily broken up into small portions.

"We conclude, therefore, that the rings must consist of disconnected particles; these may be either solid or liquid, but they must be independent. The entire system of rings must therefore consist either of a series of many concentric rings, each moving with its own velocity, and having its own system of waves, or else of a confused multitude of revolving particles, not arranged in rings, and continually coming into collision with each other."

It is true, that LAPLACE, in his investigations " On the figure of the Rings of Saturn," after showing, by the same kind of reasoning which he has already used in determining the figures of the Earth and Jupiter, that its transverse sections must be ellipses, in order that " an infinitely thin stratum of fluid, spread upon the surface of the ring, would be in equilibrium by means of the forces acting upon it," makes the hypothesis of a homogeneous fluid ring, and shows that the form of its sections must also be elliptical, in order that particles may be retained on its surface. He also shows that its time of rotation is the same as that of a satellite which should move in the path described by the centre of the generating ellipse of the ring, which ellipse must " vary in magnitude and position throughout the whole extent of the generating circumference of the ring, as such inequalities are necessary to maintain the ring in its equilibrium about Saturn." But from this it is by no means to be inferred that LAPLACE supposed that they are really fluid, or that there is, in the nature of the case, any necessity for such an hypothesis. For he distinctly says, "that the smallness of the width and thickness of any one of the rings, in comparison with its distance from the centre of Saturn, seems to increase the accuracy of the application of the preceding theory to the figure of the ring; and to render more probable the explanation we have given of the manner in which it can be sustained about the planet, by the laws of the equilibrium of fluids."

We have deemed the above exposition of LAPLACE'S views necessary, as some, from the perusal of the very able and valuable work on the " History of Physical Astronomy," by ROBERT GRANT, Esq., have supposed that LAPLACE was the first to perceive the necessity of the hypothesis of fluidity.

We have now laid before our readers the results arrived at by those who have most carefully studied this deeply interesting subject. It will be observed that Mr. BOND'S argument is based upon the fact, that the conditions of stability of a single solid homogeneous ring of uniform dimensions, or a number of such concentric rings, are inconsistent with the observed dimensions of the ring, and that if they are supposed to be irregular solids, then they must almost inevitably destroy each other. Professors PEIRCE and MAXWELL, however, base their argument upon the mechanical conditions involved in such a system. We have stated the results at which they have arrived with sufficient fulness to show in what respect they agree, and to what extent they differ.

On HANSEN's *Lunar Theory.* By A. CAYLEY, Esq, F. R. S., Quarterly Mathematical Journal, Vol. I., pp. 112–125. (1855.)

A Memoir on the Problem of Disturbed Elliptic Motion. By A. CAYLEY, Esq., F. R. S. Read March 9, 1858, before the *Royal Astronomical Society.*

On the Development of the Disturbing Function in the Lunar Theory. By A. CAYLEY, Esq., F. R. S. Read November 12, 1858, before the *Royal Astronomical Society.*

We have brought these papers together, because they have the same general aim, namely, the elucidation and more systematic development of HANSEN's Lunar Theory, as given in his *Fundamenta Nova;* and because the whole subject is now in a much more intelligible and accessible form. The whole argument, as given in these papers, is full and complete; and the reader, with the proper knowledge of analysis, will have no difficulty in understanding it. It will be remembered, that, in the problem of disturbed elliptic motion, the longitudes are measured on the varying plane of the orbit; and the position of this varying plane is determined by reference to the varying plane of the Sun's orbit, and one of the serious difficulties of the problem has been, to take these variations into account; and indeed we do not think that the author states the case too strongly, when he says, " that in memoirs and works on the Lunar and Planetary Theories, it is often difficult to discover where or how (or whether at all) account is taken of these variations." HANSEN has treated the problem rigorously. His system of differential equations involves seven arbitrary constants; six of them define the position of the body with reference to fixed axes, and the seventh determines the position of the arbitrary origin in the orbit from which the longitudes in orbit are counted.

This origin, Mr. CAYLEY calls the " Departure-point; " and longitudes in orbit counted from this point, he calls " Departures." If the node be referred to the departure-point, the seventh constant may be taken as the departure of the node. Now the problem of disturbed elliptic motion is to find the variations of the arbitrary constants in terms of the partial derivatives of the disturbing function, and the departure of the node becomes variable like the other elements.

The departure-point in this case is no longer fixed; but its locus is an orthogonal trajectory of the successive positions of the plane of the orbit; that is, the variation of the longitude of the node projected on the plane of the orbit gives the variation of the departure of the node.

The longitude of the node, the departure of the node, and the inclination, form a group of three elements which fix the position of the plane of the orbit and the departure-point; and two additional elements, the radius vector and departure, give the position of the planet in its orbit. But these last two may be expressed in terms of four others, the mean distance, the mean anomaly, the eccentricity, and the departure of the pericentre. We have then a group of five elements, or of seven; and we may suppose these elements to vary with reference to a fixed plane and a fixed origin of longitudes in this plane; or this plane of reference and origin of longitudes may both be variable, which will give rise to a new source of variation in the first three of these groups of elements. The author has given the formulæ for the variations of these groups of elements when the orbit is referred to a fixed plane, and also when it is referred to a variable plane. When the plane of reference is variable, the origin of longitudes, or departure-point, in this plane, is determined in precisely the same way that it is in the variable plane of the orbit; that is, by the intersections of an orthogonal trajectory of all its successive positions.

After giving the formulæ for the variations of the arbitrary constants, the author proceeds in his next memoir, to the development of the perturbative function, which he effects by HANSEN's method. All the terms of the development are given in an explicit form, and then compared with the developments of LUBBOCK and PONTECOULANT. Only a single inaccuracy was

found in HANSEN's development. A term in the evection should be $-\frac{3}{2}e + \frac{2}{2}\frac{3}{4}e^3 +$ &c., and not $-\frac{3}{2}e + \frac{3}{2}\frac{3}{4}e^3 +$ &c., as HANSEN has it. It is desirable to determine to what extent this error affects the moon's place as given by HANSEN's tables.

The form in which the coefficients of the different terms of the development, is given, is admirable on every account. It is better for taking the partial derivatives; and for the numerical reductions, the advantage is equally decided. We regret that want of space prevents our giving a fuller digest of these truly valuable papers, which we heartily commend to the attention of our readers.

A New Method for Correcting a Planet's Orbit. By TRUMAN HENRY SAFFORD, A. B. (Published in the Memoirs of the American Academy of Arts and Sciences, Vol. VI. New Series.)

The method here given is somewhat similar to the one in "Theoria Motus," Art. 188; the difference being, that GAUSS's method is one of false position, and Mr. SAFFORD's is differential in its character. The advantage of the differential method is, that it not only shortens the calculations, but enables the computer to dispense altogether with seven figure logarithms, except, perhaps, when the series of observations used possesses uncommon accuracy. Large portions of the computations are made with four and five figure logarithms. The application of the differential method for correcting the orbit is not new; but, as heretofore used, does not seem to have materially shortened the labor. In the above method, however, length of computation is avoided, first by a new set of geocentric coördinates, longitudes and latitudes, referred not to the plane of the Earth's orbit, but to the plane of the approximate orbit already known, of the planet in question; second, by the use of a set of heliocentric differential coördinates, as they may be called, analogous to those which Prof. HANSEN has employed so successfully in the theory of perturbations. This memoir shows that HANSEN's method of representing the effects of small variations in the elements of the orbit upon the heliocentric place is applicable not only to the problem of disturbed elliptic motion, but also to that of pure elliptic motion.

Researches in the Higher Algebra: A Paper by JAMES COCKLE, M.A., F. R. A. S.; and read by Rev. ROBERT HARLEY, F. R. A. S., October 5th, 1858, before the Literary and Philosophical Society of Manchester, England.

The following abstract has been communicated to the Monthly: —

" The author, after adverting to the complexity of the results of the higher algebra, proceeds to simplify some of them. For this purpose he employs a set of canonical functions of the unreal fifth roots of unity, and a certain system of six-valued functions of the roots of an equation of the fifth degree. Availing himself of one of the trinomial forms to which Mr. JERRARD and Sir W. R. HAMILTON have shown that the general quintic may be reduced, he has, by an indirect process, succeeded in obtaining the actual expression for the equation of the sixth degree to which that system leads. The resulting sextic is of a simple, and, viewed by the light of Mr. JERRARD's dicoveries, of a comparatively general form. So that the paper may be considered as presenting, on the one hand, the type of a class of equations of the sixth degree, whose finite algebraic solution may be effected by means of one of the fifth, or, on the other hand, as offering a resultant of the sixth degree, the simplicity of which may remove obstacles to the discussion of its solvability. Under the latter aspect the author suggests that his final sextic may perhaps throw light upon the question of the solvability of others, which occur in the theory of quintics."

" In a postscript to the above paper, dated September 10th, 1858, the author indicates the paths which may be pursued in ulterior investigations. He states that Mr. HARLEY, in some as

yet unpublished labors, has verified several of the coefficients of the equation in θ, and introduced improvements into the general theory."

" In a second postscript, dated September 22d, 1858, the author points out that the general solution of a given equation of the fifth degree may be made to depend upon that of the equation in θ."

The Method of Symmetric Products, and its Application to the Finite Algebraic Solution of Equations: A Paper by Rev. ROBERT HARLEY, F. R. A. S., read April 5th, 1859, before the Literary and Philosophical Society of Manchester, England.

The following abstract has been communicated to the Monthly : —

" This Paper is divided into three sections. The first contains a systematic exposition of Mr. COCKLE'S Method of Symmetric Products, with illustrations of its power and efficiency when applied to the lower equations. In the second, the Author discusses the resolvent product (θ) for quintics, and defines a new cyclical symbol (Σ'). He shows that θ has six, and only six, values, and that, when any one of these values vanishes, the equation of the fifth degree admits of finite algebraic solution : its roots are actually exhibited. Mr. COCKLE'S new solvable form is verified, and shown to include, as particular cases, the quadrinomials of DE MOIVRE and EULER. The third section contains a direct calculation of the equation in θ. The coefficients are followed, one by one ; the calculation being carried on by means of the cyclical symbol Σ', which is shown to possess peculiar working properties. The resulting sextic is found to coincide with Mr. COCKLE'S equation, obtained by a wholly different method, which was laid before the Society a few months ago in his ' Researches in the Higher Algebra.' The author notices the steadiness with which the Method of Symmetric Products mounts up to the higher equations, and concludes by expressing his belief that the equation in θ — the verification of which has involved prodigious labor — will be found to be a canonical equation in the theory of quintics."

Rational Cosmology : or the Eternal Principles and the Necessary Laws of the Universe. By LAURENS P. HICKOK, D. D., Union College, Schenectady, N. Y. New York : D. Appleton and Co. 1858.

From the last paragraph on page iv. of the first number of the Mathematical Monthly, I infer it to be the intention of the editor occasionally to notice and review such new mathematical publications as may from time to time appear. I would respectfully ask your attention to the above work. The high standing of the author, and the well-merited reputation he has acquired for his previous philosophical productions, are such as cannot fail to insure a favorable reception to this latest and *favorite* (as I am assured) effort of his energetic pen. But as the title of the present work is not mathematical, it is not directly adapted to mathematical readers, and thus might not receive that prompt criticism and exposure which its interference deserves. With an obviously superficial knowledge of the principles and conditions of demonstration requisite in the treatment of questions of mechanical and physical science, and with an evident foregone conclusion in his mind, and, as is easily obtained from a slight reading of very elementary treatises, our author rushes to the end of his demonstration with a rapidity truly alarming to the cautious reasoner, and apparently unconscious that a *non-sequitur* is grinning at his heels. If his book shall be suffered to remain before the public without the application of a little wholesome mathematical logic by way of a corrective, it may, from the conceded ability and widespread reputation of its author, seconded by the captivating generality and pliant subserviency of the hypothesis which he has made the basis of all his deductions (?), tend to impose for a time upon many merely literary or even *quasi*-scientific readers.

But a greater evil may result, if the errors in detail, of which two thirds of this volume are

made up, are not timely condemned by a mathematical tribunal. If such astonishing oversights are committed in the treatment of mechanical and physical questions, where the data are furnished us from without by the hand of nature, how far is there safety in relying upon the treatment of moral questions, in which the data and the method of reasoning are both at the entire disposition of the reasoner? His psychological and moral writings have obtained a favorable verdict: Behold now his long-promised theory of physics! Shall this last be allowed to act to the discredit of the former? No! it is only another instance in evidence, that the human mind of itself alone, unaided by the coöperation of material forces and the observation of material phenomena, is totally unable to divine the laws of the phenomona of the universe, or to discover the principles that determine the fulfilment of these laws. Just as the substance water is the resultant of the two component forces which manifest themselves separately in the form of oxygen and hydrogen, so man's knowledge is the resultant given by the combination of material with spiritual forces, and not by the exercise alone of spiritual force upon inactive or plastic matter.

From among the numerous examples of inconsequent reasoning which fill the greater part of the pretentious volume in question, the following may be selected as one of sufficient prominence to deserve attention. After arriving, by a very peculiar route, at the conclusion, that the force of gravitation decreases in energy according to the inverse square of the distance from the attracting centre (in which remarkable deduction, by the way, the two different relations of *difference* and *quotient* are confounded in the one term *inverse*), the demonstration of the law of falling bodies is taken up at page 157 of the Cosmology. Under the avowed hypothesis, that the energy of the attractive force increases with the increase of the inverse square of the distance from the centre, the demonstration obviously starts with the supposition that such energy increases uniformly with the decrease of the distance to the centre; while the results reached, as expressed in the uniform increase of the velocity of the falling body and the distance described by the fall, are the results due to the action of a *constant* force, such as is the force of gravitation throughout a small change of distance towards the centre. This conclusion the author was bound to reach, in order to avoid any discrepancy that might arise between the *rational* demonstration and the well-known mathematical and experimental ones; his result is safe, but the haphazard method of reaching it is open to grave objection. The accompanying figure is given to facilitate the application of numerical computation to a few steps of the author's verbal deductions.

C being the attracting centre and O the position of the body destined to describe a distance increasing as the square of the time of its fall, the crosses mark infinitesimal units of distance $1\,l$, and the right-hand figures denote the corresponding energy of the central force. For if, as the Cosmology states, the energy be called 1 at the point O, and become 2 at the point so marked, it must increase by unity for each unit of approach towards the centre, and so become 3, 4, 5, &c., at the points thus reached. By virtue of the attracting force $1\,\phi$ at the point O, the body falls through the distance $1\,l$ in the first moment of time: at the point 2, the accession of energy is $2\,\phi$, and, the former energy being retained, the sum $3\,\phi$ carries three steps to the point 5, during the second moment of time, making the distance $4\,l$ from the origin O. So far appears to be plain enough; but now, at the point 5, the accession of energy is evidently $5\,\phi$, and in addition there is on hand also the accessions $3\,\phi$ and $4\,\phi$ received at their respective points, making in all the sum $(5+4+3+2+1)\,\varphi = 15\,\varphi$, with which energy the body must proceed during the third moment of its fall, which is an increase of rapidity not at all contemplated at the outset; since the distance described in the moment should really be $5\,l$, making

the distance 9 l from the origin at the expiration of that time. True, the demonstrator quotes the number 2 as the measure of the accession of velocity received by the falling body at the end of each successive moment of time ; but this flatly contradicts the hypothesis of an increased energy of the force during the fall, and places the question within a province of reasoning, with which the rational cosmologist is obviously but superficially acquainted.

Faults of reasoning and misapprehension of principles equally gross, appear in nearly every example given in this systematic attempt to detail the application of the *absolutely* philosophical method to the constructive demonstration of the facts and phenomena of the natural world. A reference to the discussion of the second and third laws of motion, at pages 122 and 126, may please the mathematical tyro ; while physicians and astronomers will doubtless be mutually edified by the author's theories of terrestrial magnetism, and of the formation of the galaxy, &c. &c.

θ.

Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the March number of the MONTHLY : —

GEORGE B. HICKS, Student, Cleveland, Ohio, answered questions II., IV., and V.

ASHER B. EVANS, Junior Class, Madison University, Hamilton, N. Y., answered questions I., III., and IV.

If it is the opinion of teachers that the Prize Problems we select are too difficult, we shall be very much obliged if they will send us such as they think suitable. We are anxious not to defeat the desired end by giving problems either too easy or too difficult. . . . . . At a meeting of the Board of Trustees of Columbia College, held the 1st Monday in May, Professor CHARLES DAVIES was appointed Professor of Higher Mathematics, and WILLIAM G. PECK, Professor of Mathematics. . . . . . In the Note on page 279, we referred to two solutions of a problem on tangent circles in GILL's *Mathematical Miscellany;* but did not at the time notice that it also contains interesting solutions by Professors G. B. DOCHARTY, MARCUS CATLIN, BENJAMIN PEIRCE, C. GILL, and T. STRONG, of other special cases of the problem on the tangency of circles. . . . . . Professor HOYT writes : " In the April No., p. 231, line 11, for ' A. D.' read ' that line.' The perpendicular is not represented in the figure. The mode of division is essentially the same as the second one described on pages 159, 160." It is proper to state that Professor HOYT's manuscript was received before the issue of the February No., containing the construction to which he refers. . . . . . In the May No., Prize Problem II., for " Transpose " read " Transform ;" on page 263, line —3, for " R " read " K ;" on page 282, line —12, for " Art. 1135," read " Art. 1535." . . . . . We must beg of our correspondents to send us plain manuscript, and carefully drawn cuts of the right size ; so that they can be put into the hands of the printer and engraver just as we receive them.

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